Gravitation Theory in Path Space*,**

Claudio Teitelboim

Centro de Estudios Científicos de Santiago, Casilla 16443, Santiago 9,
and Departamento de Matemáticas, Facultad de Ciencias, Universidad de Chile, Santiago, Chile
and Institute for Advanced Study, Princeton, NJ 08540, USA

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A formulation of gravitation theory originally proposed by Mandelstam is re-examined. The idea is to avoid the use of coordinates while staying in the continuum. This is accomplished by regarding a point as the end of a path. The theory is then formulated in the space of all paths. The analysis relies on the properties of path deformations. These deformations play the role of gauge transformations in path space. Their algebra is established. It closes if and only if the defining conditions of a riemannian geometry hold (Bianchi identity and vanishing of the antisymmetric part of the Riemann tensor in three of its indices). Two problems faced by Mandelstam are solved: (i) An explicit formula is given which establishes when two neighboring paths end at the same point, (ii) An action principle is given, in terms of a functional integral over path space. It is also indicated how to reconstruct the metric from the curvature through gauge fixing in path space. Brief comments are offered on the possibility of developing an invariant description of loops regarded as boundaries of two-dimensional surfaces.

1. Introduction

Physical theories of fundamental significance tend to be gauge theories. These are theories in which the physical system being dealt with is described by more variables than there are physically independent degrees of freedom. The physically meaningful degrees of freedom then re-emerge as being those invariant under a transformation connecting the variables (gauge transformation). Thus, one introduces extra variables to make the description more transparent and brings in at the same time a gauge symmetry to extract the physically relevant content.

It is a remarkable occurrence that the road to progress has invariably been toward enlarging the number of variables and introducing a more powerful symmetry rather than conversely aiming at reducing the number of variables and eliminating the symmetry.

In general relativity one normally describes the field by giving the elements of the matrix that takes the tangent basis of a coordinate system onto an orthonormal basis ("Vierbein"). Both the coordinate system and the orthonormal basis are arbitrary. In d dimensions the gauge transformations are parametrized by d functions of d variables corresponding to changes of coordinates and by $\frac{1}{2}d(d-1)$ local rotations thus adding up to $\frac{1}{2}d(d+1)$ independent gauge transformations.

There is however a different approach to gravitation theory – or perhaps one should say to riemannian geometry – that introduces an enormously larger redundancy in a highly non-trivial but quite natural way.

The approach was pioneered by Mandelstam in the early sixties [1], and it is quite attractive because it makes the concept of a point a derived one. A point is regarded as the end of a path. There are infinitely many paths leading to any given point from a specified reference point. Thus a point is a class of equivalence of paths. Physical theories are then formulated in path space. This permits to avoid introducing space-time coordinates while staying in the continuum. The price paid is the introduction of a *very* large redundancy but as we stated above this is not to be regarded in itself as a drawback.

The original papers of Mandelstam dealing with electrodynamics and gravitation were written in 1962. The electrodynamics paper introduces what we would now perhaps call a "Wilson line" (a Wilson loop would be a closed Wilson line). We will use the term "Mandelstam path" in this article. Later, in 1968, he further applied the formalism to the Yang–Mills and

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Reprint requests to Prof. Dr. C. Teitelboim, Electronic address: cecsnet@uchcecvm.bitnet.

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gravitational fields and obtained the Feynman rules for both cases. At about the same time Faddeev and Popov [2] introduced the path integral over gauge fields which provided a more efficient way of obtaining the Feynman rules and became, for this reason, the standard method. In this article we are not concerned with Feynman rules but rather with the approach itself. We combine the Mandelstam method with path integrals to write the action for the field as a functional integral. This functional integral over functions of one variable ("paths") replaces the ordinary integral over space and time in the action for a field.

In order to keep track of the redundancy inherent in the path description we introduce the idea of a path deformation. The path deformations are the gauge transformations of this approach. They obey an algebra that has interesting properties, notably it closes only when the defining conditions of a riemannian geometry (Bianchi identity, vanishing of the antisymmetric part of the Riemann tensor in three of its indices) are met.

In this approach neither the metric tensor, nor the connection ever appear explicitly (there are no coordinates!). The gravitational field or, rather, the geometry, is described directly in terms of the Riemann tensor. It is only when one fixes the gauge, by selecting a unique path that leads to each endpoint from the reference point, that a coordinate system is set up on space-time. Then one can, in principle, reconstruct the metric tensor from the curvature.

The plan of the paper is the following: Sect. 2 deals with the formulation of riemannian geometry in terms of Mandelstam paths. The notion of path deformation is introduced and a criterion for establishing when two neighboring paths end at the same point is given. The algebra of paths deformations is found. (The detailed analysis is given in appendix A.) Sect. 3 discusses how to define the action as a functional integral over path space. It is shown that the Faddeev–Popov determinant plays the role of a space-time volume element. The treatment is to be regarded as provisional, since the gauge algebra involves the fields through the Riemann tensor, and hence a proper BRST treatment [3] – which is not given – is needed. It is however verified (appendix B) that the naive treatment gives the correct answer for the simple case of a 2-sphere. Finally, appendix C discusses how to reconstruct the metric form the curvature, giving an explicit construction for Riemann normal coordinates and for spherical coordinates.

2. Riemannian Geometry in Terms of Paths

2.1. Mandelstam Paths

In a riemannian manifold we take an arbitrary reference point P_{ref}. At P_{ref} we also fix a local orthonormal frame V_{ref}. Both P_{ref} and V_{ref} are fixed once and for all. From P_{ref} we draw a curve that ends at a generic point P of the manifold. A reference frame is set at all points of the curve, and in particular at the end point P, by parallel transport of V_{ref} from P_{ref} along the curve. The curve with the frames thus set along will be called a Mandelstam path, or simply a path.

We want to specify a path without recourse to a coordinate system. This is done by giving at every point of the path the components of the tangent vector referred to the frame attached to the path. We will denote those components by $u^a(\lambda)$. The parameter λ will vary between two fixed values λ_1 and λ_2. The value $\lambda=\lambda_1$ will correspond to the reference point λ_{ref} and λ_2 to the end point. All the equations will be invariant under reparametrizations of λ.

Geometrically one constructs the path by starting at the reference point and performing an infinitesimal displacement $u^a(\lambda)\,d\lambda$ relative to the frame V_{ref}, taking the frame along by parallel transport. There, at the new point, one travels $u^a(\lambda+d\lambda)\,d\lambda$ relative to the frame thus established at that point, etc.

It is to be emphasized that in this formulation there are no arbitrary coordinates to be changed nor arbitrary frames to be rotated. These concepts are never introduced. Also, although we have used the words "parallel transport" above, the connection will never appear in the formulae. It is only to make contact with the ordinary treatment of riemannian geometry that we say "the frames that come with the paths are to be regarded as parallely transported". This defines the connection.

2.2. Path Deformations

We are interested in knowing when two paths $u_1^a(\lambda)$ and $u_2^a(\lambda)$ end at the same point. If we had a coordinate system then one would simply demand that the coordinates of the end point be the same. However due to the "non-holonomic" character (in curved space) of the intrinsic description in terms of $u^a(\lambda)$, the analysis is much more elaborate. It turns out however that one can give an explicit formula for deciding when the two paths differ infinitesimally.

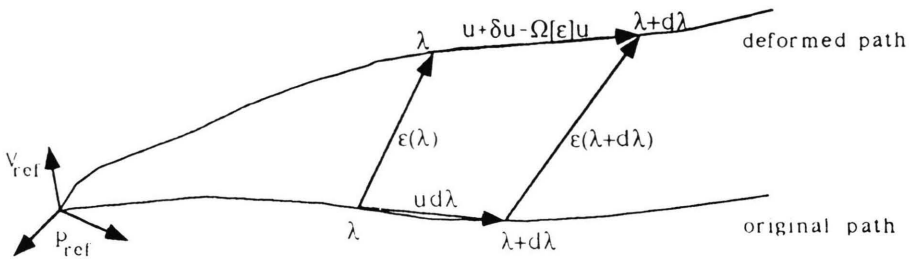

Fig. 1. *Path deformations.* The vector $\varepsilon^a(\lambda)$ connects the point with the parameter value λ on the original path with the point with the same parameter value on the deformed path. At each point of either path there is a standard orthonormal frame (not shown) obtained by parallel transport, along the path, of the reference frame at the reference point. One may parallely transport along $\varepsilon^a(\lambda)$ the standard basis at λ on the original path to the point λ on the deformed path. It will arrive rotated with respect to the standard basis at λ on the deformed path. The rotation matrix is the Ω^a_b of equation (2.1 b). This result may be obtained by dividing the region between the two paths by a sequence of infinitesimal cells of the type shown in this figure. By definition of the Riemann tensor the amount of rotation in each cell is $R^a_{bcd}\, u^c\, \varepsilon^d$.

In the closed diagram shown in the figure $u+\delta u$ is referred to the standard basis on the deformed path whereas ε, u and $\varepsilon(\lambda+\mathrm{d}\lambda)$ are referred to the standard basis on the original path. Thus to add $u+\delta u$ to ε and close the diagram one must first parallely transport it along ε to the original path. This brings in the second term on the right side of eq. (2.1 a).

Consider, as shown in Fig. 1, two neighboring paths The first path is defined by $u^a(\lambda)$ and the second $u^a(\lambda)+\delta u^a(\lambda)$. Call $\varepsilon^a(\lambda)$ the components of the vector that goes from the point with the parameter value λ on the original path to the point with the same λ on the deformed path. The components $\varepsilon^a(\lambda)$ are referred to the frame at λ on the first path. We will call $\varepsilon^a(\lambda)$ a "path deformation".

The following relation then holds:

$$\delta u^{(a)}(\lambda)=\frac{\mathrm{d}\varepsilon^a(\lambda)}{\mathrm{d}\lambda}+\Omega^a_b(\lambda)\, u^b(\lambda)\,, \tag{2.1 a}$$

$$\Omega^a_b(\lambda)=\int_{\lambda_1}^{\lambda} R^a_{bcd}(\bar{\lambda})\, u^c(\bar{\lambda})\, \varepsilon^d(\bar{\lambda})\, \mathrm{d}\bar{\lambda}\,. \tag{2.1 b}$$

We shall call (2.1) the "equation of path deformation". It is closely related to the equation of geodesic deviation of riemannian geometry. One can derive it (Fig. 1) just from the basic definition of the Riemann tensor as giving the amount of rotation after transport along an infinitesimal closed curve.

The Riemann tensor in (2.1 b) is evaluated on the first path. This means, in particular, that all the components are referred to the frame along the path obtained by parallel transport of the reference frame from the reference point. There is thus no problem in adding tensors at different points through the integral (2.1 b) because the basis at each point is fixed.

If we are given u^a and $u^a+\delta u^a$ we need to solve Eqs. (2.1) for ε^a with the initial condition $\varepsilon^a(\lambda_1)=0$. If the $\varepsilon^a(\lambda)$ thus obtained is such that $\varepsilon^a(\lambda_2)=0$ the two paths will end at the same point. It is important to realize that the criterion depends on the value of the Riemann tensor along the first path. Thus two paths which end at the same point in the presence of one curvature will not do so in general if the curvature is changed.

In flat space, where one has $R^a_{bcd}=0$ for all paths, two paths end at the same point if and only if the "total displacement"

$$x^a(\lambda_2)=\int_{\lambda_1}^{\lambda_2} u^a(\lambda)\, \mathrm{d}\lambda \tag{2.2}$$

is the same for both. The $x^a(\lambda_2)$ are then just the cartesian coordinates of the final point. However, this is not longer valid in curved space where the integral (2.2) is of no special significance.

The interest of the displacements $\varepsilon^a(\lambda_2)$ goes beyond their usefulness in determining when two paths end at the same point. In fact the functions $u^a(\lambda)$ provide a coordinate system in path space, i.e. the infinitely many numbers $u^a(\lambda)$ for $\lambda\in[\lambda_1,\lambda_2]$ and $a=1,\ldots,d$ are the coordinates of a point. The variations $\delta u^a(\lambda)$ are then the corresponding coordinate one-forms which are exact. The $\varepsilon^a(\lambda)$ defined in terms of $\delta u^a(\lambda)$ by solving Eqs. (2.1) with $\varepsilon^a(\lambda_1)=0$ are a basis of 1-forms which are not exact ("non-holonomic") but are local in space-time.

2.3. Path Derivatives

One may have functions defined over path space. These will be functionals of $u^a(\lambda)$. The values of the functionals that will be of interest in what follows may depend on the point λ along the path and also have

components referred to the frame at the point λ. They may be written as $\psi^A(\lambda)[u]$ where A is an index in some representation of the Lorentz group (vector, tensor, spinor, ...).

If one deforms the path the induced change in the field is given in terms of the functional derivative $\delta\psi/\delta u$ by

$$\delta\psi^A(\lambda)=\int_{\lambda_1}^{\lambda_2} d\bar{\lambda}\,\delta u^a(\bar{\lambda})\frac{\delta\psi^A(\lambda)}{\delta u^a(\bar{\lambda})}. \tag{2.3}$$

There are some functionals that are particularly simple and important, namely those that are obtained by evaluating a field defined over space-time on a path. They will be called local functionals. For them, the standard covariant derivative exists and may be related in a simple manner to the path derivative.

Consider, for definiteness a vector field $A^a(\lambda)[u]$. According to (2.3) its change under a path deformation is

$$\delta A^a(\lambda)=\int_{\lambda_1}^{\lambda_2}\frac{\delta A^a(\lambda)}{\delta u^b(\bar{\lambda})}\delta u^b(\bar{\lambda})\,d\bar{\lambda}. \tag{2.4}$$

But if A^a is a local field it will not be sensitive to alterations $\delta u^b(\bar{\lambda})$ for $\bar{\lambda}>\lambda$, because those will only imply changes in the path for $\lambda>\bar{\lambda}$. On the other hand $\delta A^a(\lambda)$ is sensitive to all $\delta u^a(\bar{\lambda})$ for $\bar{\lambda}<\lambda$ because those change the location in space-time of the point λ. Thus we have

$$\frac{\delta A^a(\lambda)}{\delta u^b(\bar{\lambda})}=0\quad \bar{\lambda}>\lambda, \tag{2.5}$$

and we can rewrite (2.4) as

$$\delta A^a(\lambda)=\int_{\lambda_1}^{\lambda}\frac{\delta A^a(\lambda)}{\delta u^b(\bar{\lambda})}\delta u^b(\bar{\lambda})\,d\bar{\lambda}. \tag{2.6}$$

On the other hand, from the definition of the covariant derivative and the discussion of Fig. 1, we have

$$\delta A^a(\lambda)=\varepsilon^b(\lambda)(\nabla_b A^a)(\lambda)+\Omega^a_b(\lambda)A^b(\lambda). \tag{2.7}$$

This equation may be compared with (2.6) by recalling the change of basis (2.1) that relates δu with ε. This gives

$$\varepsilon^b(\lambda)(\nabla_b A^a)(\lambda)+\Omega^a_b(\lambda)A^b(\lambda) \tag{2.8}$$
$$=\int_{\lambda_1}^{\lambda}\delta\bar{\lambda}\frac{\delta A^a(\lambda)}{\delta u^b(\bar{\lambda})}\left[\frac{d\varepsilon^b(\bar{\lambda})}{d\bar{\lambda}}+\Omega^b_m(\bar{\lambda})u^m(\bar{\lambda})\right]=\varepsilon^b(\lambda)\frac{\delta A^a(\lambda)}{\delta u^b(\lambda)}$$
$$+\int_{\lambda_1}^{\lambda}\left[\frac{d}{d\bar{\lambda}}\left(\frac{\delta A^a(\lambda)}{\delta u^b(\bar{\lambda})}\right)\varepsilon^b(\bar{\lambda})+\frac{\delta A^a(\lambda)}{\delta u^m(\bar{\lambda})}\Omega^m_n(\bar{\lambda})u^n(\bar{\lambda})\right]d\bar{\lambda}.$$

From this we infer

$$\nabla_b A^a(\lambda)=\frac{\delta A^a(\lambda)}{\delta u^b(\lambda^-)}=\lim_{\varepsilon\to 0^+}\frac{\delta A^a(\lambda)}{\delta u^b(\lambda-\varepsilon)}. \tag{2.9}$$

Note that the limit of coinciding arguments must be taken with the differentiation acting away from the point λ toward the beginning of the path. If it is taken in the opposite direction one gets zero from (2.5). Thus for space-time fields the first path derivatives are discontinuous. The amount of the jump is the covariant derivative,

$$\lim_{\lambda'\to\lambda}\frac{\delta A^a(\lambda)}{\delta u^b(\lambda')}=\lim_{\lambda'\to\lambda}h(\lambda-\lambda')\nabla_b A^a(\lambda), \tag{2.10}$$

where h is the Heaviside step function $h(\alpha)=1$ for $\alpha>0$ and $h(\alpha)=0$ for $\alpha<0$.

The relation (2.7) contains further information. Once eq. (2.10) is used one finds, by changing the order of the double integral that comes in from Ω^m_n and comparing the coefficients of $\varepsilon^b(\lambda')$ in the interval $\lambda_1<\lambda'<\lambda$

$$-\frac{d}{d\lambda'}\left(\frac{\delta A^a(\lambda)}{\delta u^b(\lambda')}\right)+\int_{\lambda'}^{\lambda}d\bar{\lambda}\,R^m_{ncb}(\lambda')u^c(\lambda')u^n(\bar{\lambda})\frac{\delta A^a(\lambda)}{\delta u^m(\bar{\lambda})}$$
$$=R^a_{mnb}(\lambda')u^n(\lambda')A^m(\lambda). \tag{2.11}$$

If we now take the limit $\lambda'\to\lambda^-$ the integral tends to zero and we find the important relation

$$-\frac{d}{d\lambda^-}\frac{\delta A^a(\lambda)}{\delta u^b(\lambda)}=R^a_{mnb}(\lambda^-)u^n(\lambda^-)A^m(\lambda). \tag{2.12}$$

Thus we can write for $\varepsilon>0$

$$\frac{\delta A^a(\lambda)}{\delta u^b(\lambda-\varepsilon)}=\nabla_b A^a(\lambda)+\varepsilon R^a_{mnb}(\lambda)u^n(\lambda)A^m(\lambda)+O(\varepsilon^2),$$
$$\frac{\delta A^a(\lambda)}{\delta u^b(\lambda+\varepsilon)}=0. \tag{2.13}$$

It is essential to keep proper track of the discontinuities in the path derivatives to avoid contradiction. For example the second covariant derivative reads

$$\nabla_c\nabla_d A^a(\lambda)=\frac{\delta}{\delta u^c(\lambda^{--})}\left(\frac{\delta A^a(\lambda)}{\delta u^d(\lambda^-)}\right). \tag{2.14}$$

Here the limits $\lambda^{--}\to\lambda$ and $\lambda^-\to\lambda$ are to be taken keeping $\lambda>\lambda^->\lambda^{--}$: For successive functional derivatives with coinciding arguments the order of the

arguments moves toward the reference point as each new derivative is taken. Thus

$$\nabla_d \nabla_c A^a(\lambda) = \frac{\delta}{\delta u^d(\lambda^{--})}\left(\frac{\delta A^a(\lambda)}{\delta u^c(\lambda^-)}\right), \tag{2.15}$$

and although second functional derivatives do commute expressions (2.14) and (2.15) are not equal. This is because they differ by the exchange of λ^- and λ^{--} and there is a discontinuity at $\lambda^- = \lambda^{--}$ whose strength is precisely the Riemann tensor.

2.4. *The Algebra of Path Deformations*

The descriptions of a path deformation by means of the non-holonomic displacement ε^a is not only local but it also has built in the geometry of space-time. This is because its relation to the holonomic basis δu^a involves the Riemann tensor components. It is therefore to be expected that the commutation law for path deformations in the non-holonomic basis will be sensitive to the properties of the Riemann tensor. This is indeed so, as we now proceed to demonstrate.

We first perform a deformation with parameter ε^a and subsequently another with parameter η^a. From that result we subtract what is obtained by acting first with η^a and next with ε^a. If in the resulting expression we only keep up to terms of order $\varepsilon\eta$, we obtain the commutator $[\eta, \varepsilon]$ of the two infinitesimal path deformations. The details of calculation are given in appendix A. The only assumption made on the Riemann tensor is that it is a local field, namely

$$\begin{aligned}\delta_\varepsilon R_{abcd} = \varepsilon^f \nabla_f R_{abcd} - \Omega^m_a R_{mbcd} \\ - \Omega^m_b R_{amcd} - \Omega^m_c R_{abmd} - \Omega^m_d R_{abcm}\,.\end{aligned} \tag{2.16}$$

It is of particular interest to see the dependence of the algebra on the defining conditions for a Riemannian geometry, that is the Bianchi identity and the vanishing of the torsion. Thus those two properties will not be assumed. The only properties assumed in addition to (2.16) are

$$R_{abcd} = -R_{bacd} \tag{2.17a}$$

and

$$R_{abcd} = -R_{abdc}\,. \tag{2.17b}$$

One finds then

$$(\delta_\eta \delta_\varepsilon - \delta_\varepsilon \delta_\eta)\, u^a = \delta_{[\eta,\varepsilon]}\, u^a + t^a + \omega^a_b\, u^b, \tag{2.18}$$

with

$$[\eta, \varepsilon]^a = \Omega^a_c[\eta]\, \varepsilon^c - \Omega^a_c[\varepsilon]\, \eta^c, \tag{2.19}$$

and

$$t^a = u^d\, \varepsilon^b\, \eta^c (R^a_{dbc} + R^a_{cdb} + R^a_{bcd}), \tag{2.20}$$

$$\omega^a_b(\lambda) = -\int_{\lambda_1}^{\lambda} u^d\, \varepsilon^p\, \eta^c (\nabla_c R^a_{bdp} + \nabla_p R^a_{bcd} + \nabla_d R^a_{bpc}). \tag{2.21}$$

The functional $\Omega^a_c[\varepsilon]$ appearing in (2.19) is the rotation (2.1 b). As illustrated in Fig. 2 the commutator (2.19) has a simple geometrical interpretation in terms of the "failure of the parallelogram to close due to relative rotation".

We see from (2.18) that the algebra only closes when one has the vanishing of the totally antisymmetric part

$$R^a_{dbc} + R^a_{cd}\, db + R^a_{bcd} = 0 \tag{2.21}$$

(vanishing of the covariant exterior derivative of the torsion two-form) and

$$\nabla_c R^a_{bdp} + \nabla_p R^a_{bcd} + \nabla_d R^a_{bpc} = 0 \tag{2.22}$$

(vanishing of the covariant exterior derivative of the curvature two-form: Bianchi Identity).

Thus the algebra of path deformations closes if, and only if, the defining conditions of a riemannian geometry hold.

This is a satisfying result. If one were to do a BRST analysis of the path gauge invariance one should find that the square of the BRST charge is zero if an only if (2.21) and (2.22) hold. Thus one would find that the "vanishing of the boundary of the boundary" is contained in BRST invariance. It is hoped to analyze this issue in detail elsewhere.

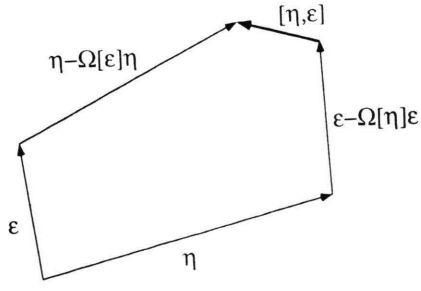

Fig. 2. *Commutation of path deformations.* If one performs first a deformation ε and afterwards a deformation η one does not find the same result as when the order is reversed. This is again due to the fact that one must correct (as in Fig. 1) for the relative rotation of the corresponding basis.

3. The Action in Path Space

3.1. Action as Functional Integral

One would like to formulate in path space the action principle for a field theory on a riemannian manifold, including general relativity. Such a formulation would provide a coordinate-free treatment while staying in the continuum.

In [1] it was found that while it was simple to find action densities it was not clear how to integrate them. The reason was that one wanted to integrate over space points regarded as end of paths, but is was not known when two paths ended at the same point.

This difficulty can be overcome now because functional integral techniques for gauge fields have become available. Thus, we will integrate over all paths $u^a(\lambda)$ treating

$$\delta u^a = \frac{\mathrm{d}\varepsilon^a}{\mathrm{d}\lambda} + \Omega^a_b\, u^b, \tag{3.1 a}$$

with

$$\varepsilon^a(\lambda_1) = \varepsilon^a(\lambda_2) = 0 \tag{3.1 b}$$

as a gauge symmetry.

In taking into account the invariance (3.1) we will use the Faddeev–Popov prescription. This is to be regarded only as a provisional treatment – perhaps not generally valid ("generally" meaning for all Riemann tensors and all gauge choices) – because the algebra (2.19) has field dependent structure constants (Ω^a_b depends on the Riemann tensor). A proper treatment should be based on a BRST analysis [3] of (3.1). As already expressed, we hope to address that problem elsewhere. In the simple example discussed in appendix B the Faddeev–Popov prescription gives the correct answer.

The action is written as a functional integral over all paths

$$S = \int \mathscr{L}\, \mathscr{D}u. \tag{3.2}$$

For the measure over path space we will take the obvious choice – consistent with local Lorentz invariance

$$\mathscr{D}u = \prod_{a,\lambda} \mathrm{d}u^a(\lambda)\, \delta[C]\, \det \mathscr{M}_C\,. \tag{3.3}$$

Here C is a gauge condition that, once the end point is fixed, picks a unique path leading to it from the reference point. The symbol $\det \mathscr{M}_C$ stands for the associated Faddeev–Popov determinant. One may, also, replace $\delta[C]$ by a more general "averaging" gauge fixing functional.

The action density $\mathscr{L}$ is to be invariant under (3.1). This is easily accomplished by demanding $\mathscr{L}$ to be a Lorentz invariant local field evaluated at the endpoint. Indeed if $\mathscr{L}$ is a local field and one deforms the path keeping the endpoint fixed its value at the endpoint will only change by a Lorentz rotation. If $\mathscr{L}$ is Lorentz invariant the rotation will have no effect.

Thus for a scalar field we have

$$\mathscr{L} = -\frac{1}{2}\eta^{ab}\frac{\delta\phi(\lambda)}{\delta u^a(\lambda^-)}\frac{\delta\phi}{\delta u^b(\lambda^-)}, \tag{3.4}$$

and for a Dirac field

$$\mathscr{L} = \bar{\Psi}\left(\gamma^a \frac{\delta\psi(\lambda)}{\delta u^a(\lambda^-)} + m\Psi\right). \tag{3.5}$$

For the gravitational field one takes the components R^a_{bcd} themselves as the basic field variables, subject to the constraints (2.17), (2.21) and (2.22) and sets

$$\mathscr{L} = R^{ab}_{ab}\,. \tag{3.6}$$

Note that the Faddeev–Popov determinant, which depends on R^a_{bcd}, plays the role of $\sqrt{g}$ in the volume element.

3.2. Gauge Fixing

One must write down a condition that picks a unique representative within the class of equivalence of all paths that end at a given end-point and can be continuously deformed into each other. In order for the condition to be a good one the (regularized) Faddeev–Popov operator M_C defined by

$$M_C\,\varepsilon = \delta_\varepsilon\, C, \tag{3.7}$$

must be invertible in the space

$$\varepsilon^a(\lambda_1) = \varepsilon^a(\lambda_2) = 0\,. \tag{3.8}$$

3.2.1. Riemann normal coordinates. A simple choice is

$$C^a = \frac{\mathrm{d}u^a}{\mathrm{d}\lambda}\,. \tag{3.9}$$

The equation $C=0$ says that u^a is parallely transported and thus the path chosen is a geodesic that goes from the reference point to the arbitrary endpoint. One may view the constants of integration $u^a(\lambda_1)$ as the space-time coordinates of the endpoint in the coordinate system implied by $\mathrm{d}u^a/\mathrm{d}\lambda = 0$.

Once the factor $\delta[C]$ is taken into account, the integral over all functions $u^a(\lambda)$ reduces to an integral over $u^a(\lambda_1)$, the action then reads

$$S=\int \mathscr{L}(\det M_C)\prod_a \mathrm{d}u^a(\lambda), \tag{3.10}$$

which shows very clearly that det M_C is $\sqrt{g}$ in the coordinate system defined by (3.9). The Faddeev–Popov operator obtained by evaluating the change of (3.9) under a path deformation reads, when $\mathrm{d}u/\mathrm{d}\lambda=0$

$$(M_C\,\varepsilon)=\frac{\mathrm{d}^2\varepsilon^a}{\mathrm{d}\lambda^2}+R^a_{bcd}\,u^c\,\varepsilon^d. \tag{3.11}$$

One knows that, for example, on a sphere the Riemann normal coordinates fail at the antipodal point of the reference point. This will show itself in (3.11) failing to be invertible for that particular value of $u^c(\lambda_1)$. However, from the point of view of the integration in (3.10) that failure is not to be regarded as more worrisome that the failure of the ordinary polar coordinates at the origin. The case of the two-dimensional sphere is treated in detail in appendix B.

3.2.2. Spherical coordinates. Another useful gauge is obtained by dividing the total interval $[\lambda_1, \lambda_2]$ in d subintervals (d is the space-time dimension) and fixing the gauge as follows

$$\begin{aligned}
&\text{1st interval:} && \frac{\mathrm{d}u^1}{\mathrm{d}\lambda}=u^2 && =\ldots && =0,\\
&\text{2nd interval:} && u^1 && =\frac{\mathrm{d}u^2}{\mathrm{d}\lambda} && =\ldots && =0,\\
&d\text{th interval:} && u^1 && =\ldots && =\frac{\mathrm{d}u^d}{\mathrm{d}\lambda} && =0.
\end{aligned} \tag{3.12}$$

The d integration constants $x^1=u^1$ (1st interval); $x^2=u^2$ (2nd interval), …, $x^d=u^d$ (dth interval) are then the coordinates of the endpoint in the gauge (3.12). In flat space these are again cartesian coordinates – and coincide in that case with Riemann normal coordinates. For a sphere they are spherical coordinates (reference point on the equator, u^1 proportional to the azimuth).

The Faddeev–Popov operator is obtained by appropriately folding the subintervals as explained in detail for the case of the 2-sphere in appendix B.

4. Added Note. Loops as Boundaries

This article has dealt with the invariant description of points regarded as boundaries of paths. It is also an attractive possibility to consider instead loops (closed paths) as the fundamental objects of interest. The question then arises as to whether one can describe a loop without introducing coordinates. As we now briefly explain, for a path homotopic to a point, this can be done by a simple extension of the ideas discussed above. For more general paths it is not clear how to proceed.

The main idea is to consider a loop that can be deformed to a point as the boundary of a disk, or, rather, as the common boundary of all the elements of a family of disks. This is implemented as follows. One deals with loops that go through a reference point. The reference point itself is identified with the zero loop whose equation is

$$u_{(0)}(\lambda)=0. \tag{4.1}$$

Next, one considers a one-parameter family of closed paths, labeled by a parameter τ with $\tau_1\leqslant\tau\leqslant\tau_2$. This family is defined by giving the deformation vectors $\varepsilon^a(\lambda,\tau)$ that take the path τ to the path with $\tau+\mathrm{d}\tau$. The value τ_1 corresponds to the zero loop and hence one sets

$$\varepsilon^a(\lambda,\tau_1)=0. \tag{4.2}$$

The value τ_2 corresponds to the loop one wishes to describe. One is sure that each path in the family is closed by demanding

$$\varepsilon^a(\lambda_1,\tau)=\varepsilon^a(\lambda_2,\tau)=0. \tag{4.3}$$

The sequence of deformations $\varepsilon^a(\lambda,\tau)$ for $\tau_1\leqslant\tau\leqslant\tau_2$ defines a disk. At each point of the disk the vectors u^a and ε^a are tangent to it and span an area element $\sigma^{ab}=u^a\varepsilon^b-u^b\varepsilon^a$. There is a whole class of equivalence of disk whose boundary is the loop in question. One may write a formula analogous to (2.1) relating the tangent vectors of two neighboring equivalent disks. That formula defines a gauge transformation in the space of disks. A theory in the space of loops is then a theory in the space of disks which is invariant under the gauge transformation.

The problem with this treatment of loops is that it is not clear how to deal with loops that are not homotopic to the zero loop. One might at first think that it would suffice to simply declare that there is a family of basic loops $u_{(0)}(\lambda)$, $u_{(1)}(\lambda)$, … which are given to start with as not homotopic to each other. One would then consider deformations away from each of the basic loops (and not just away from $u_{(0)}$). However one faces here a serious difficulty. It is the following. If $u_{(i)}(\lambda)$

describes a closed path for a given Riemann tensor it will fail to do so if the curvature is altered, even slightly. For this reason we regard a satisfactory description of loops along these lines as not yet available.

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Appendix A. Path Deformation Algebra

In this appendix we derive (3.2) by direct calculation. It will only be assumed that R_{abcd} is a local field obeying

$$R_{abcd} = -R_{bacd} = R_{badc}\,. \tag{A.1}$$

Since it is necessary to keep track of the dependence of the rotation Ω^a_b on both u and ε we will write

$$\Omega^a_b(\lambda)\,[u,\varepsilon] = \int_{\lambda_1}^{\lambda} R^a_{bcd}(\bar\lambda)\,[u]\,u^c(\bar\lambda)\,\varepsilon^d(\bar\lambda)\,\mathrm{d}\bar\lambda\,. \tag{A.1}$$

Whenever the dependence on either u or ε is irrelevant that argument will just be omitted. Similarly with λ.

We start with a path u and apply first a deformation ε that takes it onto u'. Then we apply a second deformation η which takes u' into u''. Since we are interested in the commutator we will only keep terms up to the bilinear order $\varepsilon\eta$ and, further, we will only be interested in the part antisymmetric under exchange of ε and η.

If we denote by an overdot a derivative with respect to λ, we have from the basic path deformation relation (2.1)

$$\begin{aligned} u''^a &= u'^a + \dot\eta^a + \Omega^a_b[u',\eta]\,u'^b \qquad\qquad (A.2)\\ &= u^a + \dot\varepsilon^a + \Omega^a_b[u,\varepsilon]\,u^b + \dot\eta^a \\ &\quad + \Omega^a_b[u+\dot\varepsilon+\Omega[u,\varepsilon]\,u,\eta]\,(u+\dot\varepsilon+\Omega[u,\varepsilon]\,u). \end{aligned}$$

If we now substract from (A.2) the same expression with ε and η interchanged we find

$$\begin{aligned} [\delta_\eta\,\delta_\varepsilon - \delta_\varepsilon\,\delta_\eta]\,u^a &= \Omega^a_b[u,\eta]\,(\dot\varepsilon^b + \Omega^b_c[u,\varepsilon]\,u^c) \qquad (A.3)\\ &\quad + (\delta_\varepsilon\,\Omega^a_b[u,\eta])\,u^b - (\varepsilon \leftrightarrow \eta)\,. \end{aligned}$$

Here

$$\delta_\varepsilon\,\Omega^a_b[u,\eta] \equiv \int_{\lambda_1}^{\lambda} [\delta_\varepsilon(R^a_{bcd}\,u^c)]\,\eta^d\,. \tag{A.4}$$

Now, the first term on the left side of (A.3) may be rewritten, after antisymmetrization, as

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\Omega^a_b[\eta]\,\varepsilon^b - \Omega^a_b[\varepsilon]\,\eta^b]) + u^d\,\varepsilon^b\,\eta^c\,(R^a_{bcd} + R^a_{cdb}), \tag{A.5}$$

which suggests that the commutator yields a new deformation whose parameter is

$$[\eta,\varepsilon]^a = \Omega^a_b[\eta]\,\varepsilon^b - \Omega^a_b[\varepsilon]\,\eta^b\,. \tag{A.6}$$

It is necessary to show that the other terms in (A.3) follow suite.

We have

$$\begin{aligned} \delta_\varepsilon\,\Omega^a_b[u,\eta] &= \int_{\lambda_1}^{\lambda} \{R^a_{bcd}\,\dot\varepsilon^c + \varepsilon^p\,\nabla_p\,R^a_{bcd} \qquad (A.7)\\ &\quad + \Omega^a_m[\varepsilon]\,R^m_{bcd} - \Omega^m_b[\varepsilon]\,R^a_{mcd} - \Omega^m_d[\varepsilon]\,R^a_{bcm}\}\,u^c\,\eta^d\,. \end{aligned}$$

So that, after integrating by parts, taking into account $\varepsilon^a(\lambda_1) = \eta^a(\lambda) = 0$ and rescaling that for a space-time field $(\mathrm{d}/\mathrm{d}\lambda) = u^p\,\nabla_p$, we find

$$\begin{aligned} &\delta_\varepsilon\,\Omega^a_b[\eta] - \delta_\eta\,\Omega^a_b[\varepsilon] \\ &\quad = \int_{\lambda_1}^{\lambda} (\nabla_p\,R^a_{bcd} + \nabla_c\,R^a_{bdp} + \nabla_d\,R^a_{bpc})\,u^c\,\varepsilon^p\,\eta^d \\ &\qquad + (R^a_{bcd}\,\varepsilon^c\,\eta^d)\,(\lambda) + R^2 \text{ terms}. \qquad (A.8) \end{aligned}$$

The terms not involving R^2 in (A.8) combine with the second term in (A.5) to give the contributions

$$t^a + \omega^a_b\,u^b \tag{A.9}$$

to the right-hand side of (A.3), with

$$t^a = u^d\,\varepsilon^b\,\eta^c\,(R^a_{dbc} + R^a_{cpd} + R^a_{bcd}), \tag{A.10}$$

and

$$\omega^a_b(\eta) = -\int_{\lambda_1}^{\lambda} u^d\,\varepsilon^p\,\eta^c\,(\nabla_c\,R^a_{bdp} + \nabla_p\,R^a_{bcd} + \nabla_d\,R^a_{bpc}). \tag{A.11}$$

It remains to work out the R^2 terms. They come from the Ω^2 term in (A.3) and from (A.7) and read

$$\begin{aligned} \Big[\Omega^a_p[\eta]\,\Omega^p_b[\varepsilon] + \int_{\lambda_1}^{\lambda} \{&\Omega^a_m[\varepsilon]\,R^m_{bcd} - \Omega^m_b[\varepsilon]\,R^a_{mcd} \qquad (A.12)\\ &- \Omega^m_d[\varepsilon]\,R^a_{bcm}\}\,u^c\,\eta^d\Big]\,u^b - (\eta \leftrightarrow \varepsilon). \end{aligned}$$

The coefficient of u^b in (A.12) vanishes at $\lambda = \lambda_1$. Thus it is most easily analyzed by evaluating its derivative with respect to λ and then integrating back. One thus finds that (A.12) is equal to

$$\Big[\int_{\lambda_1}^{\lambda} R^a_{bcd}\,u^c\,(\Omega^d_s[\eta]\,\varepsilon^s - \Omega^d_s[\varepsilon]\,\eta^s)\Big]\,u^b, \tag{A.13}$$

but this is just

$$\Omega^a_b[[\eta, \varepsilon]]\, u^b \tag{A.14}$$

with $[\eta, \varepsilon]$ given by eq. (A.6). This completes the calculation because we have shown

$$[\delta_\eta \delta_\varepsilon - \delta_\varepsilon \delta_\eta] u^a = \delta_{[\eta,\varepsilon]} u^a + t^a + \omega^a_b u^b \tag{A.15}$$

with $[\eta, \varepsilon]$ given by eq. (A.6) and $t^a + \omega^a_b u^b$ given by eqs. (A.10) and (A.11).

Appendix B. Faddeev–Popov Determinant for the 2-Sphere

The purpose of this appendix is to show in a simple example how the space-time volume element is recovered as a Faddeev–Popov determinant associated with gauge fixing in path space.

We consider a two-sphere with radius of curvature a. The spherical gauge (3.12) is imposed. Thus we divide the interval (λ_1, λ_2) in two subintervals $\mathrm{I} = [\lambda_1, \bar{\lambda}]$ and $\mathrm{II} = [\bar{\lambda}, \lambda_2]$. Overall normalization factors such as 2π will be systematically disregarded. However, track will be kept of the dependence on λ_1, $\bar{\lambda}$ and λ_2. Those values – being arbitrary – should drop out from the final answer. That will indeed be the case.

We denote by u_{I} the value of u^1 in the first interval and by u_{II} the value of u^2 in the second interval. Further we set

$$\omega_{\mathrm{I}} = u_{\mathrm{I}}/a,\ \omega_{\mathrm{II}} = u_{\mathrm{II}}/a \tag{B.1}$$

The Faddeev–Popov operator then reads:

Interval I:

$$(M\,\varepsilon)^1 = \ddot{\varepsilon}^1, \tag{B.1}$$

$$(M\,\varepsilon)^2 = \dot{\varepsilon}^2 + \omega_{\mathrm{I}}^2 \int_{\lambda_1}^{\bar{\lambda}} \varepsilon^2. \tag{B.3}$$

Interval II:

$$(M\,\varepsilon)^1 = \dot{\varepsilon}^1 + \omega_{\mathrm{II}}^2 \int_{\bar{\lambda}}^{\bar{\bar{\lambda}}} \varepsilon^1 - \omega_{\mathrm{I}}\,\omega_{\mathrm{II}} \int_{\lambda_1}^{\bar{\bar{\lambda}}} \varepsilon^2, \tag{B.4}$$

$$(M\,\varepsilon)^2 = \ddot{\varepsilon}^2. \tag{B.5}$$

We would like to evaluate det M by path-integral techniques. The basic formula being that the integral of the exponential of a bilinear form yields the inverse of its determinant. (We choose to integrate over ordinary commuting variables.)

The overall integral will be split in four contributions. Consider first the simplest ones, coming from ε^1 in the first interval and ε^2 in the second. One is to compute the determinant of the second derivative operator acting on functions that vanish at one end point (there are no restrictions on ε^a *at* $\bar{\lambda}$). To do so one introduces in each case an auxiliary field η and writes the quadratic form as

$$-\int_{\lambda_i}^{\lambda_f} \eta\, \dot{\varepsilon}\, \mathrm{d}\lambda. \tag{B.1}$$

This quadratic form must be rewritten as the integral of an ordinary lagrangian. To this effect one may integrate by parts to obtain

$$\int_{\lambda_i}^{\lambda_f} \dot{\eta}\, \dot{\varepsilon} \tag{B.2}$$

provided one demands

$$\eta(\lambda_i) = \eta(\lambda_f) = 0. \tag{B.3}$$

The path integral over the action (B.2) is readily evaluated by rewriting it as the transition amplitude associated with the hamiltonian

$$H = p_\varepsilon\, p_\eta \tag{B.4}$$

which comes from the lagrangian $L = \dot{\eta}\,\dot{\varepsilon}$. This automatically regularizes the determinant. One thus obtains for the path integral

$$\langle \varepsilon = \varepsilon_{\mathrm{f}}, \eta = 0 | \mathrm{e}^{-i(\lambda_b - \lambda_a) p_\varepsilon p_\eta} | \varepsilon = \varepsilon_{\mathrm{i}}, \eta = 0 \rangle, \tag{B.5}$$

where either ε_{f} or ε_{i} vanish. The amplitude (B.5) is equal to

$$\int \mathrm{d}p_\varepsilon\, \mathrm{d}p_\eta\, \mathrm{e}^{-i(\lambda_f - \lambda_i) p_\varepsilon p_\eta}\, \mathrm{e}^{-i\, p_\varepsilon (\varepsilon_{\mathrm{f}} - \varepsilon_{\mathrm{i}})} = \int \mathrm{d}p_\varepsilon\, \delta[p_\varepsilon(\lambda_f - \lambda_i)] = (\lambda_f + \lambda_i)^{-1}. \tag{B.6}$$

So coming from the (B.2) and (B.5) pieces of M we have a factor

$$(\lambda_2 - \bar{\lambda})(\bar{\lambda} - \lambda_1) \tag{B.7}$$

in the determinant.

Now, in terms of the standard spherical coordinates we have

$$u_{\mathrm{I}}(\bar{\lambda} - \lambda_1) = a\,\phi, \tag{B.8}$$

$$u_{\mathrm{II}}(\lambda_2 - \bar{\lambda}) = a\left(\frac{\pi}{2} - \theta\right). \tag{B.9}$$

[Recall that $u\,\mathrm{d}\lambda$ is a proper displacement and that the reference point is on the equator.] Thus (B.7) induces a contribution

$$a^2\, \mathrm{d}\theta\, \mathrm{d}\alpha \tag{B.10}$$

in the volume element det $M\, \mathrm{d}u_{\mathrm{I}}\, \mathrm{d}u_{\mathrm{II}}$.

Next, consider the contributions coming from (B.3) and (B.4). It is first necessary to write M as a differential operator (so that the corresponding action is the time integral of an ordinary lagrangian). This is accomplished by introducing auxiliary variables

$$\gamma^2 = \omega_{\mathrm{I}} \int_{\lambda_1}^{\bar{\lambda}} \varepsilon^2 \quad \text{(first interval)}, \tag{B.11}$$

$$\gamma^1 = \omega_{\mathrm{II}} \int_{\bar{\lambda}}^{\lambda_2} \varepsilon^1 \quad \text{(second interval)}. \tag{B.12}$$

One then integrates over both ε and γ under the constraints

$$\dot{\gamma}^2 - \omega_{\mathrm{I}} \varepsilon^1 = 0, \quad \dot{\gamma}^1 - \omega_{\mathrm{II}} \varepsilon^2 = 0. \tag{B.13}$$

The corresponding action reads

$$\begin{aligned} S = & \int_{\lambda_1}^{\bar{\lambda}} p_{\varepsilon_2}(\dot{\varepsilon}^2 + \omega_{\mathrm{I}} \gamma^2) + p_{\gamma_2}(\dot{\gamma} - \omega_{\mathrm{I}} \varepsilon^2) \\ & + \int_{\bar{\lambda}}^{\lambda_2} p_{\varepsilon_1}(\dot{\varepsilon}^1 + \omega_{\mathrm{II}} \dot{\gamma}^1) + p_{\gamma_1}(\dot{\gamma}^1 - \omega_{\mathrm{II}} \dot{\varepsilon}^1) \\ & - \omega_{\mathrm{I}} \omega_{\mathrm{II}} \left(\int_{\lambda_1}^{\bar{\lambda}} \varepsilon^2 \right) \left(\int_{\bar{\lambda}}^{\lambda_2} p_{\varepsilon_1} \right). \end{aligned} \tag{B.14}$$

The path integration is to be carried over ε^1, ε^2, γ^1, γ^2 and their conjugate momenta p_{ε_1}, p_{ε_2}, p_{γ_1}, p_{γ_2}. The integration over the p_γ's brings in an δ-functional of the constraint (B.13), whereas that over ε^1, ε^2, p_{ε_1}, p_{ε_2} yields the determinant of the operator M treated as a bilinear form. The boundary conditions on the various fields are

$$\gamma^a(\gamma_1) = \varepsilon^a(\lambda_2) = 0$$
(path fixed at both end points) (B.15)

$$\gamma^2(\lambda_1) = 0, \quad \gamma^1(\bar{\lambda}) = 0$$
(from (B.6) and (B.7)), (B.16)

$$p_{\gamma_2}(\bar{\lambda}) = p_{\lambda_1}(\lambda) =$$
(no surface terms in variation of S). (B.17)

One may treat both intervals separately keeping $\varepsilon^a(\bar{\lambda})$ fixed and then fold the partial amplitudes together by integrating their product over $\varepsilon^a(\bar{\lambda})$. Except for the last term that couples both intervals in a non-local fashion the action in each interval is in hamiltonian form with

$$H_{\mathrm{I}} = \omega_{\mathrm{I}}(p_{\varepsilon_2} \gamma^2 - p_{\gamma_2} \varepsilon^2), \tag{B.18}$$

$$H_{\mathrm{II}} = \omega_{\mathrm{II}}(p_{\varepsilon_1} \gamma^1 - p_{\gamma_1} \varepsilon^1) \tag{B.19}$$

which are nothing but rotation generators in the $(\varepsilon^2, \gamma^2)$ and $(\varepsilon^1, \gamma^1)$ planes respectively.

The non-local term can be included in the path integral over the first interval by rewriting it in the form

$$\left(-\omega_{\mathrm{II}} \int_{\bar{\lambda}}^{\lambda_2} p_{\varepsilon_1} \right) \gamma^2(\bar{\lambda}) = -P \gamma^2(\bar{\lambda}), \tag{B.20}$$

whose contribution just adds a phase in the integral over γ^2 at the joining point and replaces the condition $p_{\gamma^2}(\bar{\lambda}) = 0$ by $p_{\gamma^2}(\bar{\lambda}) = P$. Thus the path interval over the first interval yields

$$\langle \bar{\lambda} | \mathrm{e}^{-i\phi H_I} | \lambda_1 \rangle \tag{B.21}$$

with ϕ given by (B.8), and

$$|\lambda_1\rangle = |\varepsilon^2 = \gamma^2 = 0\rangle, \tag{B.22}$$

$$\langle \bar{\lambda}| = \langle \varepsilon^2 = \varepsilon^2(\bar{\lambda}), p_{\gamma_2} = P|. \tag{B.23}$$

But the state (B.22) is rotation invariant. Hence (B.21) reads imply

$$\langle \bar{\lambda} | \lambda_1 \rangle = \delta(\varepsilon^2(\bar{\lambda})), \tag{B.24}$$

and we see that P drops out.

The path integral over the second interval yields

$$\langle \lambda_2 | \mathrm{e}^{-i(\pi/2-\theta) H_{\mathrm{II}}} | \bar{\lambda} \rangle \tag{B.25}$$

with θ given by (B.9). The states in (B.25) are

$$|\langle \lambda_2| = \langle \varepsilon^1 = 0, p_{\gamma_1} = 0|, \tag{B.26}$$

$$|\bar{\lambda}\rangle = |\varepsilon^1 = \varepsilon^1(\bar{\lambda}), \gamma^1 = 0\rangle. \tag{B.27}$$

Acting on $|\bar{\lambda}\rangle$ the rotation transforms it into $|\varepsilon^1 = \sin\theta\, \varepsilon^1(\bar{\lambda}), \gamma^1 = \cos\theta\, \varepsilon^1(\bar{\lambda})\rangle$, which yields for (B.25),

$$\begin{aligned} & \langle \varepsilon^1 = 0, p_{\gamma_1} = 0 | \varepsilon^1 = \sin\theta\, \varepsilon^1(\bar{\lambda}), \cos\theta\, \varepsilon^1(\bar{\lambda}) \rangle \\ & = \delta(\sin\theta\, \varepsilon^1(\bar{\lambda})) = (\sin\theta)^{-1}\, \delta(\varepsilon^1(\bar{\lambda})). \end{aligned} \tag{B.28}$$

The inverse of the Faddeev–Popov determinant is obtained by multiplying together the partial contributions (B.7), (B.24) and (B.28) and integrating over $\varepsilon^1(\bar{\lambda})$ and $\varepsilon^2(\bar{\lambda})$. This yields for the volume element

$$\det M \, \mathrm{d}u_{\mathrm{I}} \, \mathrm{d}u_{\mathrm{II}} = a^2 \sin^2\theta \, \mathrm{d}\theta \, \mathrm{d}\phi, \tag{B.29}$$

which is the desired result.

Note that the arbitrary parameter values $\bar{\lambda}$, λ_1 λ_2 have indeed dropped out.

Appendix C. Reconstruction of the Metric from the Curvature

Suppose that one is in a particular gauge. The path leading from the reference point to a generic point will

then be specified by constants of integration $x^1, \ldots, x^d$ of the gauge conditions, where d is the space-time dimension. These x's may be thought of as the coordinates of the endpoint in that particular gauge. If we change x^a to $x^a + \mathrm{d}x^a$ there will be a unique deformation that connects the path ending at x^a with the deformed path ending at $x^a + \mathrm{d}x^a$. Let $\varepsilon^a(\lambda)$ be that deformation. The distance between x^a and $x^a + \mathrm{d}x^a$ is then

$$\mathrm{d}s^2 = \varepsilon^a(\lambda_2)\, \varepsilon^a(\lambda_2). \tag{C.1}$$

Thus, reconstructing the space-time metric amounts to solve for ε^a in terms of δu^a in the path-deformation equation (2.1). In this appendix we illustrate the procedure for the two simple examples of gauge fixing discussed in the main text.

C.1. Riemann Normal Coordinates

For Riemann normal coordinates one has

$$\delta u^a(\lambda) = \frac{\mathrm{d}\varepsilon^a}{\mathrm{d}\lambda} + \left[\int_{\lambda_1}^{\lambda} R^a_{bcd}\, u^c\, \varepsilon^d\right] u^b, \tag{C.2}$$

with $\mathrm{d}u^a/\mathrm{d}\lambda = 0$.

Eq. (C.2) may be solved for ε^a as a power series in

$$x^a = u^a(\lambda_1)\,(\lambda_2 - \lambda_1).$$

Thus, to zeroth order one has

$$\varepsilon^a(\lambda_2) = \mathrm{d}x^a, \tag{C.3}$$

and to the next order

$$\varepsilon^a(\lambda_2) = \mathrm{d}x^a - R^a_{bcd}(0)\, x^b\, x^c\, \mathrm{d}x^d. \tag{C.4}$$

So that

$$\varepsilon^a(\lambda_2)\, \varepsilon_a(\lambda_2) = (\eta_{ad} - 2R_{abcd}\, x^b\, x^c)\, \mathrm{d}x^a\, \mathrm{d}x^d, \tag{C.5}$$

and therefore

$$g_{ad} = \eta_{ad} - 2R_{abcd}(0)\, x^b\, x^c + O(x^3). \tag{C.6}$$

Thus we have obtained the standard expression for the metric tensor in Riemann normal coordinates.

C.2. Spherical Coordinates

Consider the 2-sphere in the spherical gauge already discussed in appendix B. Using the same notations as in that appendix we have that the deformation $\varepsilon^a(\lambda)$, which connects the path $(u_{\mathrm{I}}, u_{\mathrm{II}})$ with the path $(u_{\mathrm{I}} + du_{\mathrm{I}},\ u_{\mathrm{II}} + \mathrm{d}u_{\mathrm{II}})$ obeys the equations:

Interval I:

$$\mathrm{d}u_{\mathrm{I}} = \dot{\varepsilon}^1, \tag{C.7}$$

$$0 = \ddot{\varepsilon}^2 + \omega_{\mathrm{I}}^2 \int_{\lambda_1}^{\lambda} \varepsilon^2. \tag{C.8}$$

Interval II:

$$0 = \ddot{\varepsilon}^1 + \omega_{\mathrm{II}}^2 \int_{\bar{\lambda}}^{\lambda} \varepsilon^1 - \omega_{\mathrm{I}}\, \omega_{\mathrm{II}} \int_{\lambda_1}^{\bar{\lambda}} \varepsilon^2, \tag{C.9}$$

$$\mathrm{d}u_{\mathrm{II}} = \ddot{\varepsilon}^2. \tag{C.10}$$

These equations are to be solved by demanding $\varepsilon^a(\lambda_1) = 0$ and continuity at $\lambda = \bar{\lambda}$. The requirement of continuity at $\lambda = \bar{\lambda}$ is the "classical analog" of integrating over the values of the intermediate state $\varepsilon^a(\bar{\lambda})$ in the calculation of appendix B.

One finds:

Interval I:

$$\varepsilon^1(\lambda) = (\lambda - \lambda_1)\, \mathrm{d}u_{\mathrm{I}}, \tag{C.11}$$

$$\varepsilon^2(\lambda) = 0. \tag{C.12}$$

Interval II:

$$\varepsilon^1(\lambda) = (\bar{\lambda} - \lambda_1)\, \mathrm{d}u_{\mathrm{I}} \cos[u_{\mathrm{II}}(\lambda - \bar{\lambda})/a],$$

$$\varepsilon^2(\lambda) = (\lambda - \bar{\lambda})\, \mathrm{d}u_{\mathrm{II}}. \tag{C.13}$$

Therefore at the endpoint $\lambda = \lambda_2$

$$\varepsilon^1(\lambda_2) = (\bar{\lambda} - \lambda_1)\, \mathrm{d}u_{\mathrm{I}} \cos[u_{\mathrm{II}}(\lambda_2 - \bar{\lambda})/a]$$
$$= a\, \mathrm{d}\phi \sin\theta, \tag{C.14}$$

$$\varepsilon^2(\lambda_2) = (\lambda_2 - \bar{\lambda})\, \mathrm{d}u_{\mathrm{II}} = a\, \mathrm{d}\theta. \tag{C.15}$$

So that one obtains

$$\mathrm{d}s^2 = \varepsilon^a(\lambda_2)\, \varepsilon_a(\lambda_2) = a^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2), \tag{C.16}$$

the standard metric on the 2-sphere.

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